Published by Institute of Physics Publishing for SISSA/ISAS
Received: *March 6, 2003*
Accepted: *April 17, 2003*

Conformal boundary and geodesics for $AdS_5 \times S^5$ and the plane wave: their approach in the Penrose limit

Harald Dorn and Christoph Sieg

Humboldt-Universität zu Berlin, Institut für Physik
Invalidenstraße 110, D-10115 Berlin, Germany
E-mail: dorn@physik.hu-berlin.de, csieg@physik.hu-berlin.de

Abstract: Projecting on a suitable subset of coordinates, a picture is constructed in which the conformal boundary of $AdS_5 \times S^5$ and that of the plane wave resulting in the Penrose limit are located at the same line. In a second line of arguments all $AdS_5 \times S^5$ and plane wave geodesics are constructed in their integrated form. Performing the Penrose limit, the approach of null geodesics reaching the conformal boundary of $AdS_5 \times S^5$ to that of the plane wave is studied in detail. At each point these null geodesics of $AdS_5 \times S^5$ form a cone which degenerates in the limit.

Keywords: Penrose limit and pp-wave background, AdS-CFT and dS-CFT Correspondence.

Contents

1. Introduction

The AdS/CFT correspondence relates $\mathcal{N} = 4$ super Yang-Mills gauge theory in Minkowski space to type-IIB string theory in $AdS_5 \times S^5$ with some RR background flux. But since the relevant string spectrum is available only for large values of the t'Hooft coupling, explicit tests beyond the supergravity approximation remain a very difficult task. Therefore, a lot of activity has been induced by the observation of [1], that a certain sector of $\mathcal{N} = 4$ super Yang-Mills, defined by a convenient restriction to operators with a large angular momentum in S^5 , can be related to string theory in a much simpler background, namely a plane wave arising in some Penrose limit of the original background [2].

One of the most puzzling questions in this framework is the issue of holography [3]–[7]. In the standard picture the $\mathcal{N} = 4$ super Yang-Mills as the dual partner of the string theory resides on the conformal boundary of $AdS_5 \times S^5$ which is a four-dimensional Minkowski space. The plane wave is generated by zooming into the neighbourhood of a certain null geodesic in $AdS_5 \times S^5$ followed by a constant rescaling of the metric. In this procedure the old boundary is put beyond the new plane wave space. Nevertheless, string theoretic quantities referring to the new plane wave are in well established correspondence to anomalous dimensions and Green functions of a subsector of the old four-dimensional field theory [1] and [8]–[11].

Obviously, a natural starting point for re-establishing some version of the holographic picture centers around the analysis of the conformal boundary of the plane wave. Via a suitable coordinate transformation in [6] the plane wave was shown to be conformal to an Einstein static universe $R \times S^9$. The conformal boundary was found, by inspection of the singularities of the Weyl factor, to be an one-dimensional boundary line. This analysis has been confirmed also using the more general technique of terminal indecomposable pasts (futures) [12, 13].

Given these results on the conformal boundary *after* the limit has been performed, our aim in this paper is to shed some light on the behaviour of the boundary in the limiting process itself. We follow the formulation in [1], which presents the limit as a limit of the metric in given coordinates.

To this end the $AdS_5 \times S^5$ metric in global coordinates

$$ds^2 = R^2 \left(-dt^2 \cosh^2 \rho + d\rho^2 + \sinh^2 \rho \; d\Omega_3^2 + d\psi^2 \cos^2 \vartheta + d\vartheta^2 + \sin^2 \vartheta \; d\Omega_3'^{\,2}\right) \tag{1.1}$$

via the replacements

$$t = x^+ + \frac{x^-}{R^2}\,, \qquad \psi = x^+ - \frac{x^-}{R^2}\,, \qquad \rho = \frac{r}{R}\,, \qquad \vartheta = \frac{y}{R} \tag{1.2}$$

is transformed to

$$\begin{aligned} ds^2 \;=\; & R^2 (dx^+)^2 \left(\cos^2 \frac{y}{R} - \cosh^2 \frac{r}{R}\right) - 2dx^+ dx^- \left(\cos^2 \frac{y}{R} + \cosh^2 \frac{r}{R}\right) + \\ & + R^{-2} (dx^-)^2 \left(\cos^2 \frac{y}{R} - \cosh^2 \frac{r}{R}\right) + dr^2 + dy^2 + R^2 \sinh^2 \frac{r}{R} d\Omega_3^2 + R^2 \sin^2 \frac{y}{R} d\Omega_3'^{\,2}\;. \end{aligned} \tag{1.3}$$

Then in the limit $R \to \infty$ the metric becomes

$$(ds^2)_{\text{pw}} = -4dx^+ dx^- - \vec{x}^{\;2} (dx^+)^2 + (d\vec{x})^2\;. \tag{1.4}$$

Since we now have the same coordinates for the plane wave and $AdS_5 \times S^5$, we can address in section 2 the question about the relative location of the conformal boundary of both spaces. This issue will be discussed both in the coordinates just introduced as well as in the coordinates used in [6] for the identification of the conformal boundary of the plane wave.[1] To analyze the causal structure of the field theoretical holographic picture geodesics, and in particular null geodesics reaching the holographic screen out of the bulk, play a central role. With this motivation we present in sections 3 and 4 a complete classification of geodesics in $AdS_5 \times S^5$ and the plane wave (1.4) in their integrated form. Because in the chosen framework the relevant Penrose limit is realized as a limit of metrics depending on a parameter $R \to \infty$, it is obvious that locally $AdS_5 \times S^5$ geodesics converge to plane wave geodesics. But our knowledge from sections 3 and 4 will be useful for global aspects. In particular it is used in section 5 for a discussion of how differently null geodesics approach in both spaces the respective conformal boundary and in which sense the $AdS_5 \times S^5$ null geodesics reaching the $AdS_5 \times S^5$ boundary approach in the limit $R \to \infty$ null geodesics running to the plane wave conformal boundary. Finally, section 6 is devoted to a summary of the results and some conclusions.

2. Common description of conformal boundaries of $AdS_5 \times S^5$ and the BMN plane wave

Since the angular coordinate ψ in (1.1) is constrained by $-\pi \leq \psi \leq \pi$, the coordinates x^+ and x^- are constrained by

$$R^2\; x^+ - \pi R^2 \leq x^- \;\;\leq\;\; R^2\; x^+ + \pi R^2\,. \tag{2.1}$$

[1] For later reference we call them BMN and BN coordinates respectively.

JHEP04(2003)030

This is a strip in the (x^+, x^-)-plane bounded by the two parallel straight lines with slope R^2 and crossing the x^+-axis at $-\pi$ and π, respectively. For $R \to \infty$ this strip becomes the coordinate range $-\infty < x^- < \infty\,, \quad -\pi \leq x^+ \leq \pi$. Taking the limit for the metric, the identification of the two boundaries of the strip is given up, and it makes sense to extend to the whole (x^+, x^-)-plane. If one wants to avoid the restriction to the strip already for finite R, one has to puncture S^5 at its poles and to go then to the universal covering obtained by allowing ψ to take any real value.

The sequence of coordinate transformations, done in [6] to analyze the conformal boundary of the plane wave geometry (1.4), can be summarized as follows. Writing $(d\vec{x})^2 = x^2 d\Omega_7^2$, the Ω_7 coordinates remain untouched. Then in a first step one transforms[2] in the patch $x^+ \in (-\pi/2, \pi/2)$ the coordinates x^+, x^-, x to θ, φ, ζ

$$\begin{aligned} \cot\theta &= \frac{((1-x^2)\tan x^+ - 4x^-)\cos x^+}{2x}\,, \\ \tan\frac{\varphi \pm \zeta}{2} &= \frac{1}{2}(1+x^2)\tan x^+ + 2x^- \ \pm\ \frac{x}{\sin\theta\cos x^+}\,. \end{aligned} \tag{2.2}$$

The new coordinates are constrained by $0 \leq \theta \leq \pi\,,\ 0 \leq \zeta \leq \pi\,,\ |\varphi \pm \zeta| \leq \pi$. The second step uses the periodicity properties of the trigonometric functions to glue the other x^+-strips resulting in the final coordinate range

$$0 \leq \theta \leq \pi\,, \qquad 0 \leq \zeta \leq \pi\,, \qquad -\infty < \varphi < \infty\,. \tag{2.3}$$

Then the plane wave metric, up to a conformal factor, turns out to be that of the Einstein static universe $R \times S^9$. The analysis of singularities of the conformal factor, determining the conformal boundary of the plane wave, becomes most transparent after a change of parametrization of S^9. Let denote $z_1, z_2, \vec{z}$ cartesian coordinates in an embedding R^{10}, then the parametrization by θ, ζ is related to that by α, β via [3]

$$\begin{aligned} z_1 &= \cos\zeta = \sin\alpha\ \cos\beta\,, \\ z_2 &= \cos\theta\ \sin\zeta = \sin\alpha\ \sin\beta\,, \\ |\vec{z}| &= \sin\theta\ \sin\zeta = \cos\alpha\,. \end{aligned} \tag{2.4}$$

The range for α, β is

$$0 \leq \alpha \leq \frac{\pi}{2}\,, \qquad 0 \leq \beta \leq 2\pi\,. \tag{2.5}$$

Now the plane wave metric in these BN coordinates takes the form [6]

$$(ds^2)_{\rm pw} = \frac{1}{|e^{i\varphi} + \sin\alpha\ e^{i\beta}|^2}\left(-d\varphi^2 + d\alpha^2 + \sin^2\alpha\ d\beta^2 + \cos^2\alpha\ d\Omega_7^2\right). \tag{2.6}$$

The conformal factor is singular iff $\alpha = \pi/2$ and $\varphi = \beta + (2k+1)\pi\,, k \in \mathbb{Z}$. Since at $\alpha = \pi/2$ the S^7 part due the $\cos^2\alpha$ factor in front of $d\Omega_7^2$ shrinks to a point, the conformal boundary[4] of the plane wave is one-dimensional, see also figure 1. To avoid confusion in

[2]Note that our definitions for $x^\pm$ follow [1] and thus slightly differ from [6].

[3]We shift α to $\alpha - \frac{\pi}{2}$ and β to $\beta - \pi$ relative to [6].

[4]Here and for the $AdS_5 \times S^5$ case below, while speaking about the conformal boundary, we omit the two isolated points for timelike infinity.

JHEP04(2003)030

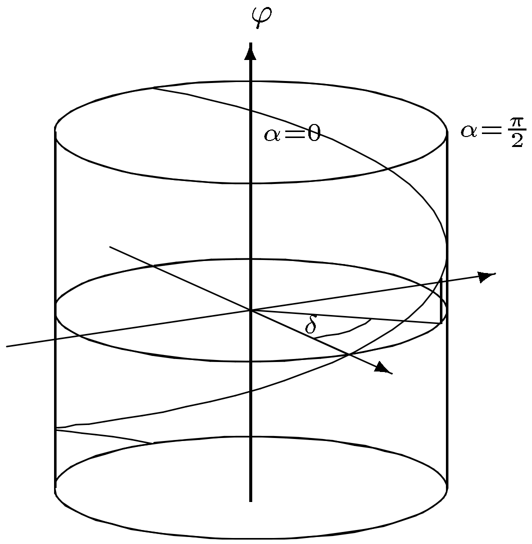

Figure 1: Part of the boundary of $AdS_5 \times S^5$ and of the plane wave in BN coordinates ($\delta = \beta + \pi$)

comparing figure 1 with similar looking pictures for $AdS_5 \times S^5$, where the half of some Einstein static universe is depicted, it is appropriate to stress that figure 1 represents the whole Einstein static universe $R \times S^9$ although the radius variable of the cylinder runs from zero to $\pi/2$ only. This range for α is due to its special role in the parametrization of S^9 in (2.4).

The coordinate transformations just discussed for the identification of the conformal boundary of the plane wave of course can also be applied to the $AdS_5 \times S^5$ metric. A priori these new coordinates are not a favourite choice to give any special insight into the $AdS_5 \times S^5$ geometry. In particular they are not well suited to find the conformal boundary.

But we can turn the argument around. Since we know already the conformal boundary of $AdS_5 \times S^5$, we can look where this boundary is situated in the new coordinates and hope to find some illuminating picture for its degeneration in the $R \to \infty$ limit which produces the plane wave metric.

Starting with global coordinates (1.1), the conformal $AdS_5 \times S^5$ boundary is at $\rho \to \infty$ with all other coordinates kept fixed at arbitrary finite values. Translating this into the coordinates of (1.3) it is at $r \to \infty$ and $x^+, x^-, y, \Omega_3, \Omega_3'$ fixed at arbitrary finite values. Before applying (2.2) we define x for the $AdS_5 \times S^5$ case by

$$r = x \cos\omega\,, \qquad y = x \sin\omega\,, \tag{2.7}$$

i.e. $x^2 = r^2 + y^2$. In the following coordinate transformation according to (2.2) $\omega, \Omega_3, \Omega_3'$ remain untouched. The conformal $AdS_5 \times S^5$ boundary is now at $x \to \infty$, $\omega \to 0$ and $x^+, x^-, \Omega_3, \Omega_3'$ fixed at arbitrary finite values. Then from the first equation (2.2) one finds (as above we again start with $|x^+| \leq \frac{\pi}{2}$ and glue the other x^+ patches afterwards)

$$\lim_{x\to\infty,\ x^+>0} \theta = \pi\,, \qquad \lim_{x\to\infty,\ x^+<0} \theta = 0\,, \qquad \lim_{x\to\infty,\ x^+=0} \theta = \frac{\pi}{2}\,. \tag{2.8}$$

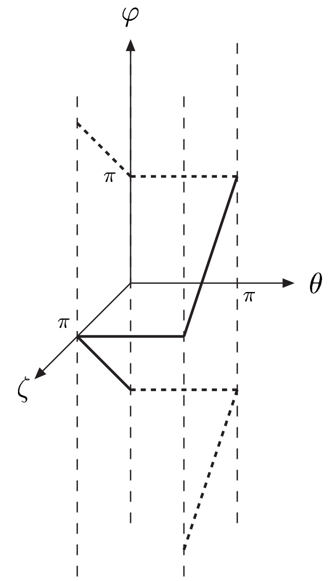

Figure 2: Part of the boundary of $AdS_5 \times S^5$ and of the plane wave in (φ, ζ, θ) coordinates.

Furthermore, by coupling $x^+ \to 0$ in a suitable way with $x \to \infty$ one can reach any $\theta \in (0, \pi)$

$$\lim_{x\to\infty,\ x^+=c/x} \theta = \arctan(-2c^{-1})\,. \tag{2.9}$$

The second equation of (2.2) then yields

$$x \to \infty\,, \quad \left\{ \begin{array}{ll} x^+ < 0 & \text{(i.e. } \theta \to 0) \\ x^+ > 0 & \text{(i.e. } \theta \to \pi) \end{array} \right\} \tag{2.10}$$

$$\Longrightarrow \quad \tan\left(\frac{\varphi+\zeta}{2}\right) \to \left\{ \begin{array}{c} \text{finite} \\ \infty \end{array} \right\}, \quad \tan\left(\frac{\varphi-\zeta}{2}\right) \to \left\{ \begin{array}{c} \infty \\ \text{finite} \end{array} \right\}. \tag{2.11}$$

In addition one gets for $x \to \infty$ coupled as in (2.9) with $x^+ \to 0$

$$x \to \infty\,, \qquad 0 < \theta < \pi \qquad \Longrightarrow \qquad \tan\left(\frac{\varphi \pm \zeta}{2}\right) \to \pm\infty\,. \tag{2.12}$$

Putting together (2.8)–(2.12), we see that in the projection onto the three coordinates φ, θ, ζ the conformal boundary of the $(|x^+| < \pi/2)$-patch of $AdS_5 \times S^5$ is mapped to the one-dimensional line starting at $(\varphi, \zeta, \theta) = (-\pi, 0, 0)$, running first with $\theta = 0$ and $\zeta - \varphi = \pi$ to $(\varphi, \zeta, \theta) = (0, \pi, 0)$, then with $\varphi = 0$ and $\zeta = \pi$ to $(\varphi, \zeta, \theta) = (0, \pi, \pi)$ and finally with $\theta = \pi$ and $\varphi + \zeta = \pi$ to $(\varphi, \zeta, \theta) = (\pi, 0, \pi)$, see also figure 2.

Translating this via (2.4) into the coordinates $\varphi,\ \alpha,\ \beta$ we find the line[5] $\alpha = \pi/2$, $\beta = \pi + \varphi,\ -\pi < \varphi < +\pi$. After gluing the other x^+-patches we can conclude: the

[5]Note that the piece from $(\varphi, \zeta, \theta) = (0, \pi, 0)$ to $(\varphi, \zeta, \theta) = (0, \pi, \pi)$ with $\varphi = 0$ and $\zeta = \pi$ is mapped to one point $(\varphi, \alpha, \beta) = (0, \frac{\pi}{2}, \pi)$.

projection onto the coordinates (φ, α, β) of the conformal boundary of $AdS_5 \times S^5$ coincides with that of a part of the conformal boundary of the plane wave (1.4). That in this projection only a part of the plane wave boundary line appears as the $AdS_5 \times S^5$ boundary is due to the restriction to the $AdS_5 \times S^5$-strip (2.1). Note that this restriction can be circumvented as discussed at the beginning of this section.

Taking into account the other seven coordinates, the $AdS_5 \times S^5$ boundary is of course not one-dimensional. But by using the same coordinates both for $AdS_5 \times S^5$ *and* the plane wave, we now have visualized the degeneration of the conformal boundary in the process of approaching the plane wave limit. In the projection to three of the BN coordinates $(\varphi,\ \alpha,\ \beta)$ the boundary stays throughout this process at the same location. The extension in the remaining coordinates degenerates to a point in the limit.

The picture is more involved if one compares the two boundaries in the BMN coordinates (x^+, x^-, x). As noted in [6], due to the singularity of the coordinate transformation on the boundary line in (φ, α, β), the limits in (x^+, x^-, x) which map to this boundary line are not unique. Besides

$$\text{Limit (i):} \qquad x \ \to\ \infty\,, \qquad x^+,\ x^- \quad \text{finite}\,, \tag{2.13}$$

just discussed above, the second limit is

$$\text{Limit (ii):} \qquad x^- \ \to\ \pm\ \infty\,, \qquad x^+,\ x \quad \text{finite}\,, \tag{2.14}$$

as trivially seen from (2.2).

Now the situation looks a little bit cumbersome. The conformal boundary of $AdS_5 \times S^5$ is realized via limit (i), that of the plane wave via both limits (i) and (ii), although the the plane wave itself is a limit of $AdS_5 \times S^5$.

To get a better understanding of this situation, we are now asking what happens in the $R \to \infty$ limit with geodesics, in particular null geodesics, which reach the conformal $AdS_5 \times S^5$ boundary. After an explicit construction of all the geodesics both for $AdS_5 \times S^5$ and the plane wave (1.4) in the next two sections, we come back to this question in section 5.

3. Geodesics in $AdS_5 \times S^5$

We start with the $AdS_5 \times S^5$ metric in global coordinates

$$ds^2 = R^2(-dt^2 \cosh^2\rho + d\rho^2 + \sinh^2\rho\ d\Omega_3^2 + d\Omega_5^2)\,. \tag{3.1}$$

The geodesic equations for the AdS_5 and S^5 coordinates decouple. Geodesics on S^5 are great circles. Whether the geodesic in the total manifold $AdS_5 \times S^5$ moves in S^5 or stays at a fixed S^5 position has consequences for the overall causal property (space-like, time-like, null) only. There is no effect on the AdS_5 coordinates. Therefore we can concentrate on the AdS_5 part. Then the geodesic equations for t, ρ appearing in (3.1) explicitly, as well as for $\phi^i,\ i = 1,2,3$ being orthogonal coordinates on the 3-dim unit sphere Ω_3, are

$$\ddot{t} + 2\dot{\rho}\ \dot{t}\tanh\rho\, = 0\,, \tag{3.2}$$

$$\ddot{\rho} + (\dot{t}\dot{t} - \dot{\phi}_i\dot{\phi}_i h_i)\sinh\rho\cosh\rho\, = 0\,, \tag{3.3}$$

$$\ddot{\phi}^i + 2\dot{\phi}^i\dot{\rho}\coth\rho + \dot{\phi}^k\dot{\phi}^j\gamma^i_{kj}\, = 0\,. \tag{3.4}$$

The dot indicates differentiation with respect to an affine parameter τ. h_i denotes the diagonal entries of the Ω_3 metric and the γ^i_{kj} are the related Levi-Civita connection coefficients.

Since the radius of the S^3 contribution in (3.1) depends on ρ, the separation of the movement in S^3 is not as trivial as the separation of the S^5 movement. However, the ρ-dependence in eq. (3.4) appears as a factor of $\dot{\phi}_i$ only. This means that this equation describes within Ω_3 a geodesic, but our parameter τ, defined to be an affine one with respect to the geodesic as a whole, is not affine for the Ω_3 part treated separately on its own. Defining a new parameter $\sigma = f(\tau)$ as a solution of the differential equation

$$\ddot{f} + 2\dot{\rho}\dot{f}\coth\rho = 0\,, \tag{3.5}$$

eq. (3.4) becomes equivalent to (the prime stands for $d/d\sigma$)

$$\phi^{''i} + \phi^{'k}\phi^{'j}\gamma^i_{kj} = 0\,. \tag{3.6}$$

Now we are sure that the length of the tangential vector $\phi^{'i}$ in terms of the 3-dim unit sphere metric is constant, i.e. $\phi^{'i}\phi^{'i}h_i = c_3 \geq 0$. Transforming this back to $\dot{\phi}^i$ we get

$$\dot{\phi}^i\dot{\phi}^i h_i = c_3\,\dot{f}^2\,, \qquad c_3 \geq 0\,. \tag{3.7}$$

The freedom to multiply an affine parameter by a constant can be used to choose $c_3 = 1$ or $c_3 = 0$ (for $\phi^i = \text{const}$). However, in eq. (3.7) the option $c_3 = 0$ is no longer necessary, since constant ϕ^i can be realized via $\dot{f} = 0$. Therefore, from now on we put $c_3 = 1$ for all cases. Inserting then (3.7) into (3.3) we arrive at

$$\ddot{\rho} + (\dot{t}\dot{t} - \dot{f}\dot{f})\sinh\rho\cosh\rho = 0\,, \tag{3.8}$$

and see that the movement in ρ and t is influenced by a possible movement within the unit 3-sphere coordinates of AdS_5 only via one function $f(\tau)$ which on its own couples back to ρ via (3.5).

Summarizing the discussion so far, the coordinates in Ω_3 either remain constant ($\dot{f} = 0$) or describe a movement on a great circle ($\dot{f} \neq 0$). In addition, the system of five coupled equations for the coordinates $\rho, t, \phi^1, \phi^2, \phi^3$ is reduced to the three coupled equations (3.2), (3.8) and (3.5) for ρ, t, f.

Straightforward integration of (3.2) and (3.5) yields

$$\dot{t} = \frac{b}{\cosh^2\rho}\,, \qquad \dot{f} = \frac{\tilde{b}}{\sinh^2\rho}\,, \qquad b,\ \tilde{b} \quad \text{constant}\,. \tag{3.9}$$

With (3.8) this gives an equation for ρ alone

$$\ddot{\rho} + \frac{b^2}{\cosh^3\rho}\sinh\rho - \frac{\tilde{b}^2}{\sinh^3\rho}\cosh\rho = 0\,. \tag{3.10}$$

Instead of solving this equation directly, we found it more convenient to make a small detour. Since our parameter τ is an affine one, the scalar product of the tangential vector with itself in the sense of the AdS metric is a constant along the geodesic, we call it c_5. Then using (3.7) and (3.9) we get

$$\dot{\rho}^2 = \frac{c_5}{R^2} - \frac{\tilde{b}^2}{\sinh^2\rho} + \frac{b^2}{\cosh^2\rho}\,. \tag{3.11}$$

Are all solutions of (3.11) also solutions of (3.10)? At first the constancy of the scalar product of the tangential vector with itself is of course a much weaker condition than the geodesic equations. But in writing down (3.11) we already have implied the geodesic equations for all coordinates, except for ρ. Under these circumstances, at least as long as $\dot{\rho} \neq 0$, the constant scalar product condition is equivalent to the geodesic equation for the last coordinate ρ.

Since $\dot{\rho}^2$ is a non-negative quantity, from (3.11) and

$$\frac{c_5}{R^2} - \frac{\tilde{b}^2}{\sinh^2\rho} + \frac{b^2}{\cosh^2\rho} \;\leq\; \frac{c_5}{R^2} + b^2\,, \qquad \forall\rho\,,$$

as a byproduct, we find a constraint on c_5 and the integration constant b

$$\frac{c_5}{R^2} + b^2 \;\geq\; 0\,. \tag{3.12}$$

For further analyzing the consequences of the positiveness of both sides of eq. (3.11) we introduce the abbreviations

$$A = \frac{c_5}{R^2}\,, \qquad B = b^2 + \frac{c_5}{R^2} - \tilde{b}^2\,, \qquad C = -\tilde{b}^2\,. \tag{3.13}$$

Then first of all, by these definitions and the inequality (3.12) the constants A, B, C are universally constrained by

$$C \;\leq\; 0\,, \qquad B \;\geq\; A + C\,, \qquad B \;\geq\; C\,. \tag{3.14}$$

In addition, checking whether there are real ρ-values for which the r.h.s. of (3.11) is non-negative, it turns out that only four classes of ranges[6] of the constants A, B, C are allowed. Integrating case by case first (3.11) and then (3.9) for the four classes one finds:
type I

$$\begin{aligned} &A > 0\,,\\ &0 \leq \frac{\sqrt{B^2-4AC}-B}{2A} \;\leq\; \sinh^2\rho\,, \end{aligned} \tag{3.15}$$

$$\begin{aligned} \rho &= \operatorname{arsh}\sqrt{\frac{1}{4A}\left(e^{\pm 2\sqrt{A}(\tau+\tau_0)} + (B^2-4AC)e^{\mp 2\sqrt{A}(\tau+\tau_0)}\right) - \frac{B}{2A}}\,,\\ t &= \pm\arctan\left(\frac{e^{\pm 2\sqrt{A}(\tau+\tau_0)} + 2A - B}{2b\sqrt{A}}\right) + t_0\,,\\ f &= \pm\arctan\left(\frac{e^{\pm 2\sqrt{A}(\tau+\tau_0)} - B}{2\tilde{b}\sqrt{A}}\right) + f_0\,, \end{aligned} \tag{3.16}$$

[6]The special case $A = B = C = 0$ corresponds to a point, not to a curve.

type II

$$
\begin{aligned}
&A < 0\,, \qquad B^2 - 4AC \; > \; 0\,, \qquad B \; > \; 0\,, \\
&0 \leq \frac{B - \sqrt{B^2 - 4AC}}{-2A} \; \leq \; \sinh^2\rho \; \leq \; \frac{B + \sqrt{B^2 - 4AC}}{-2A}\,, && (3.17) \\
&\rho \, = \, \mathrm{arsh}\sqrt{\frac{1}{-2A}(B \pm \sqrt{B^2 - 4AC}\,\sin(2\sqrt{-A}(\tau + \tau_0)))}\,, \\
&t \, = \, \pm\frac{1}{2}\,\mathrm{arccot}\left(\frac{2\sqrt{-A}\;b\cos(2\sqrt{-A}(\tau + \tau_0))}{\sqrt{B^2 - 4AC} \pm (B - 2A)\sin(2\sqrt{-A}(\tau + \tau_0))}\right) + t_0\,, \\
&f \, = \, \pm\frac{1}{2}\,\mathrm{arccot}\left(\frac{2\sqrt{-A}\;\tilde{b}\cos(2\sqrt{-A}(\tau + \tau_0))}{\sqrt{B^2 - 4AC} \pm B\sin(2\sqrt{-A}(\tau + \tau_0))}\right) + f_0\,, && (3.18)
\end{aligned}
$$

type III

$$
\begin{aligned}
&A = 0\,, \qquad B \; > \; 0\,, \\
&0 \; \leq \; \frac{-C}{B} \; \leq \; \sinh^2\rho\,, && (3.19) \\
&\rho \, = \, \mathrm{arsh}\sqrt{B(\tau + \tau_0)^2 - \frac{C}{B}}\,, \\
&t \, = \, \arctan\left(\frac{B(\tau + \tau_0)}{b}\right) + t_0\,, \\
&f \, = \, \arctan\left(\frac{B(\tau + \tau_0)}{\tilde{b}}\right) + f_0\,, && (3.20)
\end{aligned}
$$

type IV

$$
\begin{aligned}
&A < 0\,, \qquad B^2 - 4AC = 0\,, \qquad B \; \geq \; 0\,, \\
&\sinh^2\rho \, = \, \frac{B}{-2A}\,, && (3.21) \\
&\rho \, = \, \mathrm{arsh}\sqrt{\frac{B}{-2A}}\,, \\
&t \, = \, \sqrt{-A}\;\tau + t_0\,, \\
&f \, = \, \pm\sqrt{-A}\;\tau + f_0\,. && (3.22)
\end{aligned}
$$

Perhaps it is useful to stress, that in the absence of any movement in Ω_3, i.e. for $C = 0$, the formulas (3.16), (3.18) and (3.20) for ρ simplify to
type I with C=0

$$\rho = \mathrm{arsh}\left(\sqrt{\frac{|B|}{A}}\;|\sinh(\sqrt{A}(\tau + \tau_0'))|\right)\,, \tag{3.23}$$

type II with C=0

$$\rho = \mathrm{arsh}\left(\sqrt{\frac{B}{-A}}\;|\sin(\sqrt{-A}(\tau + \tau_0'))|\right)\,, \tag{3.24}$$

type III with C=0

$$\rho = \mathrm{arsh}\left(\sqrt{B}\;|\tau + \tau_0|\right)\,. \tag{3.25}$$

type	causal propert. w.r.t. AdS_5	causal propert. w.r.t. $AdS_5 \times S^5$	reaches conf. bound. of $AdS_5 \times S^5$
I	space-like	space-like	yes
II	time-like	all	no
III	null	null or space-like	yes
IV	time-like	all	no

Table 1: Characteristic properties of geodesics.

The $\pm$ alternative in (3.16) and (3.18) has been absorbed into the shift of the integration constant τ_0 to τ_0'.

The causal properties of the geodesics and their relation to the conformal boundary (note footnote 5) can be summarized in the table 1.

For later use it is important to stress, that null geodesics in the sense of full $AdS_5 \times S^5$ reaching the boundary have to be of *type III*. For them no movement in S^5 is allowed while a movement in S^3 is possible as long as $b^2 > \tilde{b}^2$.

4. Geodesics in the plane wave

Here the metric is

$$ds^2 = -4dx^+dx^- - \vec{x}^{\,2}(dx^+)^2 + (d\vec{x})^2\,, \tag{4.1}$$

and yields the geodesic equations

$$\ddot{x}^+ = 0\,, \tag{4.2}$$

$$\ddot{x}^- + \frac{1}{2}\,\dot{x}^+\frac{d}{d\tau}\vec{x}^{\,2} = 0\,, \tag{4.3}$$

$$\ddot{x}^i + (\dot{x}^+)^2 x^i = 0\,. \tag{4.4}$$

(4.2) implies linear dependence of x^+ on the the affine parameter τ

$$x^+ = \alpha\,\tau + x_0^+\,. \tag{4.5}$$

Obviously now the geodesics fall into two classes, *type A* with $\alpha = 0$ and *type B* with $\alpha \neq 0$.
type A

$$x^+ = \text{const}\,, \qquad x^- = \beta\tau + x_0^-\,, \qquad x^i = \gamma^i\tau + x_0^i\,. \tag{4.6}$$

The scalar product of their tangential vector with itself is given by $(\gamma^i)^2$. This implies: all *type A* geodesics are null or space-like. Space-like *type A* geodesics reach infinity in the transversal coordinates $\vec{x}$. *Type A* null geodesics are given by constant x^+ and x^i as well as x^- running between $\pm\infty$.
type B
Then we have (4.5) with $\alpha \neq 0$, and the integrations of (4.3), (4.4) yield in addition

$$x^i = \beta^i \sin\left(\alpha(\tau + \tau_0^i)\right), \tag{4.7}$$

$$x^- = \frac{1}{8}\sum_i (\beta^i)^2 \sin\left(2\alpha(\tau + \tau_0^i)\right) + \gamma\tau + x_0^-\,. \tag{4.8}$$

JHEP04(2003)030

The scalar product of the tangential vector with itself is now equal to $-4\alpha\gamma$, and we conclude: all *type B* geodesics either stay at $\vec{x}=0$ (for $\vec{\beta}=0$) or oscillate in the transversal coordinates x^i (for $\vec{\beta}\neq 0$). All space or time-like *type B* geodesics ($\gamma \neq 0$) reach $\pm\infty$ both in x^+ and x^-. *Type B* null geodesics ($\gamma = 0$) reach $\pm\infty$ *only* with respect to x^+. Furthermore, they stay at fixed $\vec{x}$ and x^- ($\vec{\beta}=0$) or oscillate both in $\vec{x}$ and x^- ($\vec{\beta}\neq 0$).

In conclusion null geodesics reaching the conformal boundary of the plane wave, see (2.13), (2.14), are necessarily of *type A*. There are no null geodesics reaching the conformal boundary within the asymptotic regime of limit (i).

Closing this section we comment on a simple discussion of the plane wave null geodesics in using the BN coordinates of (2.6). In general null geodesics are invariant under a Weyl transformation. Such a transformation only effects the choice of affine parameters along the null geodesics. Null geodesics with respect to (2.6) without the Weyl factor are given by great circles in S^9 accompanied by a compensating movement along the time-like direction φ. If we discuss S^9 as an embedding in R^{10}, reaching $\alpha = \frac{\pi}{2}$ is equivalent to reaching the (z_1, z_2)-plane. There are of course great circles within this plane. They correspond to null geodesics either winding at $\alpha = \pi/2$ in constant distance to the conformal plane wave boundary around the cylinder in figure 1 up to $\varphi \to \pm\infty$ or they wind in the orthogonal direction crossing the conformal plane wave boundary. In the sense of $R \times S^9$ there is nothing special with such a crossing. But going back to the metric including the Weyl factor, starting from an inside point, the boundary is reached at infinite affine parameter. Furthermore, there are of course great circles staying completely away from the (z_1, z_2)-plane (i.e. $\alpha = \pi/2$). They correspond to null geodesics generically oscillating in $0 < \alpha < \pi/2$ and running up to $\varphi \to \pm\infty$. Finally, great circles can also intersect the (z_1, z_2)-plane. Then they correspond to null geodesics oscillating in α and touching $\alpha = \pi/2$. Obviously some of them reach the conformal boundary line of the plane wave. According to the above analysis in BMN coordinates they are of *type A*, too.

5. Conformal boundaries and geodesics

As discussed in section 3, only null-geodesics of *type III* reach the conformal boundary of $AdS_5 \times S^5$. They necessarily stay at fixed S^5-position. Translating (3.20) into the coordinates of (1.3) we get

$$\begin{aligned}
x^+ &= \frac{1}{2}\left(\arctan\left(\frac{B(\tau+\tau_0)}{b}\right) + t_0 + \psi\right),\\
x^- &= \frac{R^2}{2}\left(\arctan\left(\frac{B(\tau+\tau_0)}{b}\right) + t_0 - \psi\right),\\
r &= R\ \mathrm{arsh}\sqrt{B(\tau+\tau_0)^2 - \frac{C}{B}}\,,\\
f &= \arctan\left(\frac{B(\tau+\tau_0)}{\tilde{b}}\right) + f_0\,,\\
y &= R\vartheta\,.
\end{aligned} \tag{5.1}$$

Our goal is to find in the $R \to \infty$ limit a correspondence to null geodesics of the plane wave. Therefore, our $AdS_5 \times S^5$ geodesics have to stay at least partially within the range of finite x^+, x^-, r, y. Taking $R \to \infty$ at fixed τ would send all x^- to infinity. But of course the affine parameter itself is determined only up to a constant rescaling. Therefore, the best procedure is to eliminate the affine parameter completely.

First from (5.1) we conclude, that along the full range of a *type III* null geodesic, i.e. for $(-\infty < \tau < \infty)$, the coordinate x^+ runs within an interval of length $\frac{\pi}{2}$: $x^+ \in (\frac{1}{2}(t_0+\psi) - \frac{\pi}{4}, \frac{1}{2}(t_0+\psi) + \frac{\pi}{4})$ and x^- runs within an interval of length $\frac{\pi}{2}R^2$: $x^- \in (R^2(\frac{t_0-\psi}{2} - \frac{\pi}{4}), R^2(\frac{t_0-\psi}{2} + \frac{\pi}{4}))$. To ensure that the x^- interval for $R \to \infty$ stays at least partially within the range of finite values both endpoints of the interval have to have the opposite sign. Thus we have to restrict t_0 and ψ by

$$-\frac{\pi}{2} < t_0 - \psi < \frac{\pi}{2}. \tag{5.2}$$

In addition one has universally

$$|f(\tau = +\infty) - f(\tau = -\infty)| = \pi. \tag{5.3}$$

From (3.7) f can be understood as the angle along the great circle in S^3 on which our null geodesics is running. Therefore, for *type III* geodesics the positions for $\tau = -\infty$ and $\tau = +\infty$ within the S^3 are always antipodal to each another.[7]

After these preparations we now eliminate the affine parameter and express x^+, r and f in terms of x^- (note that for *type III* we have $A = 0$ and $B = b^2 - \tilde{b}^2$)

$$\begin{aligned}
x^+ &= \frac{x^-}{R^2} + \psi\,,\\
r &= R\,\mathrm{arsh}\sqrt{\frac{\tan^2(\frac{2x^-}{R^2} - t_0 + \psi) + \frac{\tilde{b}^2}{b^2}}{1 - \frac{\tilde{b}^2}{b^2}}}\,,\\
f &= f_0 + \arctan\left(\frac{b}{\tilde{b}}\tan\left(\frac{2x^-}{R^2} - t_0 + \psi\right)\right),\\
y &= R\vartheta\,.
\end{aligned} \tag{5.4}$$

The minimal value for r is

$$r_{\mathrm{min}} = R\,\mathrm{arsh}\left(\frac{|\frac{\tilde{b}}{b}|}{\sqrt{1 - \frac{\tilde{b}^2}{b^2}}}\right).$$

Since we insist on finite r_{min} for $R \to \infty$ we have to rescale ($r_0 = \lim_{R\to\infty} r_{\mathrm{min}}$)

$$\frac{|\tilde{b}|}{|b|} = \frac{r_0}{R}. \tag{5.5}$$

Although we have now realized finite r_{min}, the x^- value where r_{min} is reached stays finite for $R \to \infty$ only if (5.2) is replaced by the stronger rescaling condition

$$t_0 - \psi = \frac{a}{R^2}. \tag{5.6}$$

[7]In the limiting case, where the null geodesics goes through $r = 0$, f becomes a step function.

Altogether, to stay at least with part of the *type III* null geodesics within the range of finite BMN coordinates, it is mandatory to perform the rescalings (5.5), (5.6) and to keep y fixed. The remaining parameters replacing $t_0, \psi, b, \tilde{b}, f_0, \vartheta$ are ψ, a, b, r_0, f_0, y.

Considering now at fixed x^- the $R \to \infty$ limit of (5.4) one arrives at

$$\begin{aligned} x^+ &= \psi + O\left(\frac{1}{R^2}\right), \\ r &= r_0 + O\left(\frac{1}{R^2}\right), \\ f &= f_0 + O\left(\frac{1}{R^2}\right), \\ y &= \text{const}\,. \end{aligned} \tag{5.7}$$

Constant r, y via (2.7) give constant x. In addition, constant f, i.e. no movement in the S^3, and the a priori absence of any movement in S^5 leads to constant $\vec{x}$. This together with the constancy of x^+ implies:

An $AdS_5 \times S^5$ null geodesics, reaching the conformal boundary, for any finite x^--interval at $R \to \infty$ converges uniformly to a *type A* null geodesics of the plane wave.

However, the approach of the $AdS_5 \times S^5$ null geodesics to the conformal boundary of $AdS_5 \times S^5$ is realized within the asymptotic regime of limit (i), see (2.13), (5.1), but that of the plane wave null geodesics within the regime of limit (ii), see (2.14) and text after (4.6). That means even for large R, after a region of convergence, on their way to the boundary they diverge from one another at the very end (in the x^+, x^-, x coordinates under discussion).

In a global setting the situation is most simply illustrated for *type III* null geodesics crossing the origin of the transverse BMN coordinates $\vec{x}$, i.e. $r_0 = y = 0$. We also put $a = 0$, the case $a \neq 0$ can be simply recovered by the replacement $x^- \to x^- - \frac{a}{2}$. Then first of all x^- runs between $\pm\,\frac{\pi}{4}\,R^2$. Furthermore, (5.4) implies ($\Theta(z)$ step function)

$$\begin{aligned} \frac{r}{R} &= F\left(\frac{x^-}{R^2}\right), \qquad \text{with} \quad F(z) = \text{arsh}|\tan(2z)|\,, \\ f &= f_0 \,\pm\, \frac{\pi}{2}\,(2\Theta(x^-) - 1)\,. \end{aligned} \tag{5.8}$$

The plane wave geodesic is at $x = y = r = 0, \quad -\infty < x^- < \infty$. It is the uniform limit for $R \to \infty$ in the region $|x^-| < R^{1-\epsilon}$. This convergence is due to the different powers of R on the l.h.s. and in the argument of the function F on the r.h.s. of (5.8), see also figure 3.

The picture in figure 3 has to be completed by the freedom to choose a point on S^3 to fix the direction in the space of the $\vec{x}$ coordinates. This completely specifies the *type III* null geodesics under discussion. Then the conformal boundary reaching null geodesics of $AdS \times S^5$ crossing the origin of the transversal BMN coordinates $\vec{x} = 0$ form a cone with base S^3. The three parameters to specify the S^3 position together with ψ nicely correspond to the four-dimensionality of the $AdS \times S^5$ boundary. In the $R \to \infty$ limit this cone degenerates.

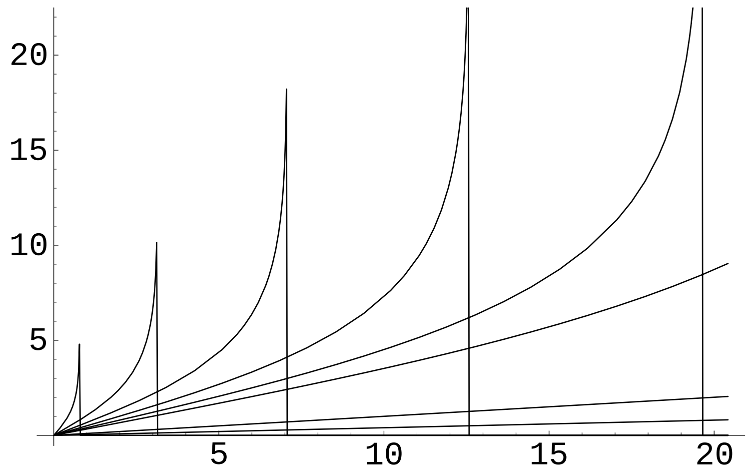

Figure 3: Approach of boundary reaching $AdS_5 \times S^5$ null geodesics to a boundary reaching null geodesics of the plane wave. The plane wave null geodesics runs along the horizontal axis up to infinity. The plot shows r versus x^- for $AdS_5 \times S^5$ null geodesics (5.8) in the cases $R = 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 20,\ 50$.

6. Conclusions

Using BN coordinates the conformal boundary of the plane wave (1.4) has been identified as an one dimensional line [6]. Drawing three of the BN coordinates (φ, α, β), the plane wave is mapped to a cylinder of infinite length and radius $\pi/2$. To any point inside this cylinder belongs a S^7. These S^7 degenerate to a point on the tube of radius $\pi/2$. The conformal boundary line spirals around the tube of radius $\pi/2$.

In this paper we have shown that using the same BN coordinates for $AdS_5 \times S^5$ its well known conformal boundary, in the projection to the three coordinates (φ, α, β), appears to be located at the same spiraling line as the conformal boundary of the plane wave. Of course for $AdS_5 \times S^5$ on this line the extension with respect to the other 7 coordinates is not degenerated to a point. But we have generated a perhaps useful intuitive picture: The boundary is always at the same line, taking the limit $R \to \infty$ the extension in the remaining 7 coordinates shrinks to a point. [8]

Switching from BN coordinates back to BMN coordinates, it turned out that, due to the singularity of the coordinate transformation at the boundary line, the approach to this line is realized within two different asymptotic regimes of the BMN coordinates, called (i) and (ii) in (2.13) and (2.14). Only one of these regimes corresponds to the conformal boundary of $AdS_5 \times S^5$.

Insight into the causal relations between bulk and boundary in the limiting process can be obtained by analyzing the behaviour of geodesics reaching the respective boundary.

We have given a complete classification for *all* geodesics, both for the original full $AdS_5 \times S^5$ and the plane wave. This classification is based on different ranges for three

[8] This then implies also the degeneration of the 3 remaining dimensions of the conformal boundary of $AdS_5 \times S^5$.

specific integration constants. Among the $AdS_5 \times S^5$ null geodesics only those of *type III* reach the conformal boundary. In BMN coordinates this approach is within the asymptotic regime (i). In contrast, null geodesics of the plane wave reaching the conformal plane wave boundary approach its boundary within regime (ii). First of all this obviously implies that for $R \to \infty$ the convergence of $AdS_5 \times S^5$ geodesics to plane wave geodesics cannot be uniform. Using our explicit formulas for the geodesics we were able to discuss the issue of convergence in more detail.

The convergence is uniform in the region $|x^-| < R^{1-\epsilon}$. Beyond this region, at any fixed R, the null geodesics of $AdS_5 \times S^5$ and the plane wave diverge while approaching the respective conformal boundary. Then the plane wave null geodesics runs up to $|x^-| \to \infty$ but stays at finite values for the remaining coordinates. On the other side, the $AdS_5 \times S^5$ null geodesics asymptotes to $x^- = \pm\frac{\pi}{4}R^2$ while some of the coordinates different from x^+ and x^- diverge. The fact of converging geodesics within $|x^-| < R^{1-\epsilon}$ fits into the naive picture that in BMN coordinates the $AdS_5 \times S^5$ space up to the order of magnitude of R looks like a plane wave.

The analysis of global properties of the null geodesics of $AdS_5 \times S^5$ crossing the origin of the transversal part of the BMN coordinates, given at the end of section 5, can be straightforwardly generalized to geodesics passing this origin at a nonzero distance. Therefore, we conclude that at each point with finite BMN coordinates the null geodesics of $AdS_5 \times S^5$ reaching the conformal boundary form a three-dimensional cone. For $R \to \infty$, in the range where the BMN coordinates stay fixed or grow slower than R, this cone degenerates to the single plane wave null geodesic crossing the point under consideration and reaching the plane wave conformal boundary. Therefore all points in this range effectively notice a degeneration of the boundary.

Acknowledgments

This work is supported by DFG (German Science Foundation) within the "Schwerpunktprogramm Stringtheorie" and the "Graduiertenkolleg 271". We thank N. Beisert, G. Curio, N. Prezas and M. Salizzoni for useful discussions.

References

[1] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and PP waves from $N = 4$ super Yang-Mills*, *J. High Energy Phys.* **04** (2002) 013 [hep-th/0202021].

[2] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *Penrose limits and maximal supersymmetry*, *Class. and Quant. Grav.* **19** (2002) L87 [hep-th/0201081];
M. Blau, J. Figueroa-O'Farrill and G. Papadopoulos, *Penrose limits, supergravity and brane dynamics*, *Class. and Quant. Grav.* **19** (2002) 4753 [hep-th/0202111].

[3] S.R. Das, C. Gomez and S.-J. Rey, *Penrose limit, spontaneous symmetry breaking and holography in PP-wave background*, *Phys. Rev.* **D 66** (2002) 046002 [hep-th/0203164].

[4] E. Kiritsis and B. Pioline, *Strings in homogeneous gravitational waves and null holography*, *J. High Energy Phys.* **08** (2002) 048 [hep-th/0204004].

[5] R.G. Leigh, K. Okuyama and M. Rozali, *PP-waves and holography*, *Phys. Rev.* **D 66** (2002) 046004 [hep-th/0204026].

[6] D. Berenstein and H. Nastase, *On lightcone string field theory from super Yang-Mills and holography*, hep-th/0205048.

[7] G. Siopsis, *Holography in the Penrose limit of AdS space*, *Phys. Lett.* **B 545** (2002) 169 [hep-th/0205302];
G. Siopsis, *The Penrose limit of $AdS \times S$ space and holography*, hep-th/0212165.

[8] C. Kristjansen, J. Plefka, G.W. Semenoff and M. Staudacher, *A new double-scaling limit of $N = 4$ super Yang-Mills theory and PP-wave strings*, *Nucl. Phys.* **B 643** (2002) 3 [hep-th/0205033].

[9] D.J. Gross, A. Mikhailov and R. Roiban, *Operators with large R charge in $N = 4$ Yang-Mills theory*, *Ann. Phys. (NY)* **301** (2002) 31 [hep-th/0205066].

[10] N.R. Constable et al., *PP-wave string interactions from perturbative Yang-Mills theory*, *J. High Energy Phys.* **07** (2002) 017 [hep-th/0205089].

[11] A. Santambrogio and D. Zanon, *Exact anomalous dimensions of $N = 4$ Yang-Mills operators with large R charge*, *Phys. Lett.* **B 545** (2002) 425 [hep-th/0206079].

[12] D. Marolf and S.F. Ross, *Plane waves: to infinity and beyond!*, *Class. and Quant. Grav.* **19** (2002) 6289 [hep-th/0208197].

[13] V.E. Hubeny and M. Rangamani, *Causal structures of PP-waves*, *J. High Energy Phys.* **12** (2002) 043 [hep-th/0211195].